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Representation theorem for local operators in the space of continuous and monotone functions

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ABSTRACT

We prove that every locally defined operator mapping the space of nondecreasing continuous functions into itself is a Nemytskii composition operator.

1. Introduction

Locally defined (or local) operators acting between the spaces of measurable (or integrable) functions were considered by many authors. For more references on this theory see J. Appell and P.P. Zabrejko [1]. To define the notion of a local operator, given an interval $I \subset \mathbb{R}$ denote by $\mathcal{G} = \mathcal{G}(I)$ and $\mathcal{H} = \mathcal{H}(I)$ two classes of functions $\varphi : I \to \mathbb{R}$. A mapping $K : \mathcal{G} \to \mathcal{H}$ will be called a *locally defined operator*, briefly, a *local operator*, or more precisely a $(\mathcal{G}, \mathcal{H})$-*local operator* if for every open interval $J \subset \mathbb{R}$ and for all functions $\varphi, \psi \in \mathcal{G}$, the implication

$$\varphi|_{J\cap I} = \psi|_{J\cap I} \quad \Rightarrow \quad K(\varphi)|_{J\cap I} = K(\psi)|_{J\cap I}$$

holds true (cf. [2–6]). In the present paper we are mainly interested in such operators in the case where $\mathcal{G} = \mathcal{H}$ is the class of nondecreasing and continuous functions denoted by $CM_+(I)$.

Local operators mapping the space $C^m(I)$ of m-times continuously differentiable functions in an interval $I \subset \mathbb{R}$ into $C^0(I)$ and $C^1(I)$ (i.e., (C^m, C^0)-local operators and (C^m, C^1)-local operators) were considered in [2]. In our recent papers [3,4] we extend the main results of [2] to the space of Whitney differentiable functions. As a corollary we obtained that $K : C^0(I) \to C^0(I)$ is local iff there exists a function $h : I \times \mathbb{R} \to \mathbb{R}$ such that, for all $\varphi \in C^0(I)$,

$$K(\varphi)(x) = h\big(x, \varphi(x)\big) \quad (x \in I),$$

and h is continuous on $I \times \mathbb{R}$, that is K is a Nemytskii operator of a generator h (cf. also [2,3]). A similar result holds true for local operators $K : C^m(I) \to C^m(I)$, $m \geqslant 1$ (cf. [4,6]). However, in the case $m = 1$, the function h need not be even continuous on $I \times \mathbb{R}$. This shows that the form of a $(\mathcal{G}, \mathcal{H})$-locally defined operator strongly depends on the nature of the function spaces $\mathcal{G}$ and $\mathcal{H}$ which are its domains and ranges, respectively.

The main result of this paper says that a locally defined operator maps $CM_+(I)$ into itself if, and only if, it is a Nemytskii (superposition) operator generated by a function that is continuous in both variables and nondecreasing with respect to

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each variable (Section 3). Applying this result, in Section 4, we obtain the characterization of (CM_+, CM_-)-, (CM_-, CM_+)-, (CM_-, CM_-)-locally defined operators.

2. Auxiliary results

In the sequel, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ denote, respectively, the set of positive integers, nonnegative integers and the set of real numbers.

Let $I \subset \mathbb{R}$ be an interval. In this paper a function $f : I \to \mathbb{R}$ is called *nondecreasing* if $f(x_1) \leqslant f(x_2)$ for all $x_1, x_2 \in I$, $x_1 < x_2$ and, *increasing* if f is nondecreasing and one-to-one. A function $f : I \to \mathbb{R}$ is referred to as *nonincreasing* (*decreasing*) if $-f$ is nondecreasing (respectively, increasing).

From now on, $C^0(I)$ stands for a family of real continuous functions defined on I, and $M_+(I)$, $M_-(I)$ denote, respectively, a family of nondecreasing and nonincreasing functions $f : I \to \mathbb{R}$. We write $CM_+(I)$ for $C^0(I) \cap M_+(I)$ and $CM_-(I)$ for $C^0(I) \cap M_-(I)$.

We start with the following

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval and $x_0 \in I$ be fixed. Suppose that an operator $K : CM_+(I) \to CM_+(I)$ is locally defined. Then for all $\varphi_1, \varphi_2 \in CM_+(I)$ the following implication holds true*:

$$\varphi_1(x_0) < \varphi_2(x_0) \quad \Rightarrow \quad K(\varphi_1)(x_0) \leqslant K(\varphi_2)(x_0). \tag{1}$$

Proof. Put $a = \inf I$ and $b = \sup I$. Assume first that $x_0 \in \operatorname{int} I$ and suppose, contrary to our claim, that there exist two functions $\varphi_1, \varphi_2 \in CM_+(I)$ such that $\varphi_1(x_0) < \varphi_2(x_0)$ and $K(\varphi_1)(x_0) > K(\varphi_2)(x_0)$. By the continuity of $K(\varphi_1)$ and $K(\varphi_2)$ there exists a real r, $0 < r \leqslant \min\{\frac{x_0-a}{4}, \frac{b-x_0}{4}\}$ such that for all $x \in I$, $|x - x_0| < 2r$,

$$\left|K(\varphi_1)(x) - K(\varphi_1)(x_0)\right| < \frac{K(\varphi_1)(x_0) - K(\varphi_2)(x_0)}{2} \tag{2}$$

and

$$\left|K(\varphi_2)(x) - K(\varphi_2)(x_0)\right| < \frac{K(\varphi_1)(x_0) - K(\varphi_2)(x_0)}{2}. \tag{3}$$

Define $\gamma : I \to \mathbb{R}$ by the formula

$$\gamma(x) = \begin{cases} \varphi_1(x) & \text{for } x \in (-\infty, x_0 - r) \cap I, \\ \frac{\varphi_2(x_0+r)-\varphi_1(x_0-r)}{2r}(x - x_0 + r) + \varphi_1(x_0 - r) & \text{for } x \in [x_0 - r, x_0 + r), \\ \varphi_2(x) & \text{for } x \in [x_0 + r, \infty) \cap I. \end{cases}$$

It is obvious that $\gamma \in CM_+(I)$.

Since

$$\gamma|_{(x_0-2r,x_0-r)} = \varphi_1|_{(x_0-2r,x_0-r)}, \qquad \gamma|_{(x_0+r,x_0+2r)} = \varphi_2|_{(x_0+r,x_0+2r)}, \tag{4}$$

according to the definition of a local operator, we have

$$K(\gamma)|_{(x_0-2r,x_0-r)} = K(\varphi_1)|_{(x_0-2r,x_0-r)}, \qquad K(\gamma)|_{(x_0+r,x_0+2r)} = K(\varphi_2)|_{(x_0+r,x_0+2r)}. \tag{5}$$

Choose an arbitrary point $x_1 \in (x_0 - 2r, x_0 - r)$ and $x_2 \in (x_0 + r, x_0 + 2r)$. Of course $x_1 < x_2$ and, by (5),

$$K(\gamma)(x_1) = K(\varphi_1)(x_1), \qquad K(\gamma)(x_2) = K(\varphi_2)(x_2).$$

Hence, taking into account (2) and (3), we obtain

$$\left|K(\gamma)(x_1) - K(\varphi_1)(x_0)\right| < \frac{K(\varphi_1)(x_0) - K(\varphi_2)(x_0)}{2}$$

and

$$\left|K(\gamma)(x_2) - K(\varphi_2)(x_0)\right| < \frac{K(\varphi_1)(x_0) - K(\varphi_2)(x_0)}{2}.$$

Therefore

$$K(\varphi_1)(x_0) - K(\gamma)(x_1) < \frac{K(\varphi_1)(x_0) - K(\varphi_2)(x_0)}{2}$$

and

$$K(\gamma)(x_2) - K(\varphi_2)(x_0) < \frac{K(\varphi_1)(x_0) - K(\varphi_2)(x_0)}{2}.$$

Adding these two inequalities we get

$$K(\varphi_1)(x_0) - K(\gamma)(x_1) + K(\gamma)(x_2) - K(\varphi_2)(x_0) < K(\varphi_1)(x_0) - K(\varphi_2)(x_0),$$

whence

$$K(\gamma)(x_2) < K(\gamma)(x_1),$$

which contradicts the nondecreasing monotonicity of $K(\gamma)$. This proves that our lemma holds true for every $x_0 \in \operatorname{int} I$.

Assume now that a, the left endpoint of I, belongs to I and put $x_0 := a$. Take an arbitrary pair of functions $\varphi_1, \varphi_2 \in CM_+(I)$ and assume that $\varphi_1(x_0) < \varphi_2(x_0)$. By the continuity of φ_1, φ_2 for an arbitrary sequence $x_k \in \operatorname{int} I$ $(k \in \mathbb{N})$, convergent to x_0, there exists $k_0 \in \mathbb{N}$ such that

$$\varphi_1(x_k) < \varphi_2(x_k), \quad k \geqslant k_0,\ k \in \mathbb{N}.$$

According to what has been proved,

$$K(\varphi_1)(x_k) \leqslant K(\varphi_2)(x_k), \quad k \geqslant k_0,\ k \in \mathbb{N},$$

whence, by the continuity of $K(\varphi_1)$ and $K(\varphi_2)$ at x_0, letting $k \to \infty$, we get

$$K(\varphi_1)(x_0) \leqslant K(\varphi_2)(x_0).$$

Similarly, we can prove that (1) is fulfilled when x_0 is the right endpoint of I. This completes the proof. $\square$

Lemma 2. *Let $I \subset \mathbb{R}$ be an interval and let $(x_0, y_0) \in I \times \mathbb{R}$, $x_0 < \sup I$ be fixed. Then for every sequence $(x_k, y_k) \in I \times \mathbb{R}$ satisfying the condition*

$$x_{k+1} < x_k; \qquad y_{k+1} \leqslant y_k, \quad k \in \mathbb{N}; \qquad \lim_{k\to\infty} (x_k, y_k) = (x_0, y_0), \tag{6}$$

there exists a function $\gamma \in CM_+(I)$ such that, for all $k \in \mathbb{N}_0$,

$$\gamma(x_k) = y_k.$$

Proof. Take an arbitrary sequence $(x_k, y_k) \in I \times \mathbb{R}$, $k \in \mathbb{N}$, satisfying (6) and define a sequence of functions $\gamma_k : I \to \mathbb{R}$, $k \in \mathbb{N}$, in the following way:

$$\gamma_k(x) := \begin{cases} y_0 & \text{for } x \in (-\infty, x_0] \cap I, \\ \frac{y_k - y_0}{x_k - x_0}(x - x_0) + y_0 & \text{for } x \in (x_0, x_k], \\ \frac{y_i - y_{i-1}}{x_i - x_{i-1}}(x - x_{i-1}) + y_{i-1} & \text{for } x \in (x_i, x_{i-1}],\ i \in \{2, \ldots, k\}, \\ y_1 & \text{for } x \in (x_1, \infty) \cap I. \end{cases}$$

Let us observe that

$$\gamma_k(x_0) = y_0, \qquad \gamma_k(x_k) = y_k = \gamma_{k+l}(x_k), \quad k, l \in \mathbb{N}, \tag{7}$$

and for every $x \in I \backslash \{x_k\colon k \in \mathbb{N}_0\}$ there exists $k_0 \in \mathbb{N}$ such that

$$\gamma_k(x) = \gamma_{k_0}(x), \quad k \geqslant k_0,\ k \in \mathbb{N}. \tag{8}$$

Put

$$\gamma(x) = \lim_{k\to\infty} \gamma_k(x), \quad x \in I.$$

By (7) and (8), the function γ is well defined. Moreover, γ is nondecreasing and $\gamma(x_k) = y_k$, for all $k \in \mathbb{N}_0$. Since the sequence $(\gamma_k)_{k\in\mathbb{N}}$ tends uniformly to γ, the function γ is continuous. This completes the proof. $\square$

Similarly, we can get the following

Remark 1. If $(x_0, y_0) \in I \times \mathbb{R}$ where $x_0 > \inf I$ and $(x_k, y_k) \in I \times \mathbb{R}$ is a sequence satisfying the condition

$$x_k < x_{k+1}; \qquad y_k \leqslant y_{k+1}, \quad k \in \mathbb{N}; \qquad \lim_{k\to\infty} (x_k, y_k) = (x_0, y_0), \tag{9}$$

then there exists a function $\gamma \in CM_+(I)$ such that, for all $k \in \mathbb{N}_0$, $\gamma(x_k) = y_k$.

Lemma 3. *Let $I \subset \mathbb{R}$ and $x_0 \in I$ be fixed. Suppose that $K : CM_+(I) \to CM_+(I)$ is locally defined. Then:*

1° *for all $\psi_1, \psi_2 \in CM_+(I)$ the following implication holds*

$$\psi_1(x_0) = \psi_2(x_0) \quad \Rightarrow \quad K(\psi_1)(x_0) = K(\psi_2)(x_0); \tag{10}$$

2° *if $\varphi_k \in CM_+(I)$,*

$$\varphi_{k+1} \leqslant \varphi_k, \quad k \in \mathbb{N}, \tag{11}$$

and the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is uniformly convergent, then

$$\lim_{k\to\infty} K(\varphi_k)(x_0) = K(\varphi)(x_0), \tag{12}$$

where $\varphi := \lim_{k\to\infty} \varphi_k$.

Proof. We begin by showing that the lemma holds true in the case where $x_0 \in \operatorname{int} I$. To prove 1° take $\psi_1, \psi_2 \in CM_+(I)$ such that $\psi_1(x_0) = \psi_2(x_0)$. Then $\gamma : I \to \mathbb{R}$ defined by

$$\gamma(x) := \begin{cases} \psi_1(x) & \text{for } x \leqslant x_0,\ x \in I, \\ \psi_2(x) & \text{for } x > x_0,\ x \in I, \end{cases}$$

belongs to $CM_+(I)$. Since

$$\gamma|_{(-\infty,x_0)\cap I} = \psi_1|_{(-\infty,x_0)\cap I}, \qquad \gamma|_{(x_0,+\infty)\cap I} = \psi_2|_{(x_0,+\infty)\cap I},$$

according to the definition of a local operator, we have

$$K(\gamma)|_{(-\infty,x_0)\cap I} = K(\psi_1)|_{(-\infty,x_0)\cap I}, \qquad K(\gamma)|_{(x_0,+\infty)\cap I} = K(\psi_2)|_{(x_0,+\infty)\cap I}.$$

Hence, by the continuity of $K(\gamma)$, $K(\psi_1)$ and $K(\psi_2)$, we get

$$K(\psi_1)(x_0) = K(\gamma)(x_0) = K(\psi_2)(x_0),$$

which implies (10).

To show 2° assume that a sequence $\varphi_k \in CM_+(I)$, $k \in \mathbb{N}$, satisfying (11) is uniformly convergent to φ. Then obviously $\varphi \in CM_+(I)$.

Fix an $\varepsilon > 0$. Since all the functions $K(\varphi_k)$, $k \in \mathbb{N}$, are continuous, there exists a $\delta_k > 0$ such that the following implication holds

$$x \in I, \quad |x - x_0| < \delta_k \quad \Rightarrow \quad \bigl|K(\varphi_k)(x) - K(\varphi_k)(x_0)\bigr| < \varepsilon, \quad k \in \mathbb{N}. \tag{13}$$

Let us choose a sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$x_k - x_0 < \delta_k, \quad k \in \mathbb{N}; \qquad \lim_{k\to\infty} x_k = x_0$$

and

$$x_k > x_0, \qquad x_{k+1} < x_k, \quad k \in \mathbb{N}. \tag{14}$$

In view of (11) and (14), we have

$$\varphi_{k+1}(x_{k+1}) \leqslant \varphi_k(x_{k+1}) \leqslant \varphi_k(x_k), \quad k \in \mathbb{N},$$

which means that the sequences $(x_k)_{k\in\mathbb{N}}$ and $(\varphi_k(x_k))_{k\in\mathbb{N}}$ satisfy the conditions of Lemma 2 with $y_k = \varphi_k(x_k)$ for $k \in \mathbb{N}$, and $y_0 = \varphi(x_0)$. Thus there exists a function $\gamma \in CM_+(I)$ such that

$$\gamma(x_k) = \varphi_k(x_k), \quad k \in \mathbb{N}, \qquad \gamma(x_0) = \varphi(x_0).$$

Hence, making use of (13), we get

$$\begin{aligned}\left|K(\varphi_k)(x_0)-K(\varphi)(x_0)\right| &\leqslant \left|K(\varphi_k)(x_k)-K(\varphi_k)(x_0)\right|+\left|K(\varphi_k)(x_k)-K(\varphi)(x_0)\right| \\ &\leqslant \varepsilon+\left|K(\gamma)(x_k)-K(\gamma)(x_0)\right|.\end{aligned}$$

Since $K(\gamma)$ is continuous, this inequality implies that

$$\lim_{k\to\infty} K(\varphi_k)(x_0)=K(\varphi)(x_0),$$

that is 2° holds true. This completes the proof in the case when $x_0 \in \operatorname{int} I$.

Now assume that x_0 is the left endpoint of I.

To show (10) take an arbitrary pair of functions $\psi_1, \psi_2 \in CM_+(I)$ such that $\psi_1(x_0)=\psi_2(x_0)$. If there exists a sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$\psi_1(x_k)=\psi_2(x_k), \quad x_k>x_0,\ k\in\mathbb{N}; \qquad \lim_{k\to\infty} x_k=x_0, \tag{15}$$

then, according to the first part of the proof, $K(\psi_1)(x_k)=K(\psi_2)(x_k)$ for all $k\in\mathbb{N}$, and, by the continuity of ψ_1,ψ_2 at x_0, letting k tend to the infinity, we obtain the desired conclusion.

If (15) is not fulfilled then there exists a $\delta_0>0$ such that $\psi_1(x)\neq\psi_2(x)$ for all $x\in(x_0,x_0+\delta_0)$. There is no loss of generality in supposing that

$$\psi_1(x)<\psi_2(x), \quad x\in(x_0,x_0+\delta_0). \tag{16}$$

Assume first that ψ_1 is not constant on any right neighbourhood of x_0 for some $\delta\in(0,\delta_0)$. In this case we construct the sequence $(x_k)_{k\in\mathbb{N}}$ in the following way. Choose a point $x_1\in(x_0,x_0+\delta)$. Since $\psi_1(x_1)<\psi_2(x_1)$, the equality $\psi_1(x_0)=\psi_2(x_0)$, the monotonicity and the Darboux property of ψ_2, imply that there is $x_2\in(x_0,x_1)$ such that $\psi_2(x_2)=\psi_1(x_1)$. Similarly for $x_3\in(x_0,x_2)$, chosen arbitrarily close to x_0, we can find x_4 such that $\psi_2(x_4)=\psi_1(x_3)$. Repeating this procedure, by induction, we can construct a sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$x_k\in I, \qquad x_{k+1}<x_k, \qquad \psi_1(x_{2k-1})=\psi_2(x_{2k}), \qquad \psi_1(x_{2k+1})<\psi_2(x_{2k}), \quad k\in\mathbb{N}.$$

Put

$$y_0:=\psi_1(x_0)=\psi_2(x_0)$$

and

$$y_k:=\begin{cases}\psi_1(x_k) & \text{for odd } k,\\ \psi_2(x_k) & \text{for even } k.\end{cases}$$

Since $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ fulfil the assumption of Lemma 2, there exists a function $\gamma\in CM_+(I)$ such that

$$\gamma(x_{2k-1})=\psi_1(x_{2k-1}), \qquad \gamma(x_{2k})=\psi_2(x_{2k}), \quad k\in\mathbb{N}.$$

Now, according to what has already been proved, we get

$$K(\gamma)(x_{2k-1})=K(\psi_1)(x_{2k-1}), \qquad K(\gamma)(x_{2k})=K(\psi_2)(x_{2k}), \quad k\in\mathbb{N}.$$

Hence, by the continuity of $K(\gamma)$, $K(\psi_1)$ and $K(\psi_2)$ at x_0, letting $k\to\infty$, we get (10).

If ψ_1 is constant on $(x_0,x_0+\delta)$ and (16) holds for some $\delta\in(0,\delta_0)$ then, obviously, there exists a sequence of increasing and continuous functions $\varphi_k:[x_0,x_0+\delta]\to\mathbb{R}$, $\psi_1<\varphi_k<\psi_2$, $\varphi_{k+1}\leqslant\varphi_k$, $k\in\mathbb{N}$, uniformly convergent to ψ_1 such that

$$\varphi_k(x_0)=\psi_1(x_0)=\psi_2(x_0).$$

Hence, by the previous case, we obtain

$$K(\varphi_k)(x_0)=K(\psi_2)(x_0), \quad k\in\mathbb{N}. \tag{17}$$

Taking into account the monotonicity of $K(\varphi_k)$, $k\in\mathbb{N}$, on $(x_0,x_0+\delta)$, we get

$$K(\varphi_k)(x_0)\leqslant K(\varphi_k)(x), \quad k\in\mathbb{N},\ x\in(x_0,x_0+\delta),$$

and, consequently, by (17),

$$K(\psi_2)(x_0)\leqslant K(\varphi_k)(x). \tag{18}$$

Since $\lim_{k\to\infty}K(\varphi_k)(x)=K(\psi_1)(x)$ for every $x\in\operatorname{int} I$, we conclude that

$$K(\psi_2)(x_0) \leqslant K(\psi_1)(x), \quad x \in (x_0, x_0 + \delta),$$

whence, by the continuity of $K(\psi_1)$ at x_0, we get

$$K(\psi_2)(x_0) \leqslant K(\psi_1)(x_0).$$

On the other hand, take an arbitrary $x \in (x_0, x_0 + \delta)$. Applying Lemma 1 for x, from (16) we get

$$K(\psi_1)(x) \leqslant K(\psi_2)(x), \quad x \in (x_0, x_0 + \delta).$$

Letting $x \to x_0$, by the continuity of $K(\psi_1)$ and $K(\psi_2)$ at x_0, we get $K(\psi_1)(x_0) \leqslant K(\psi_2)(x_0)$, which together with (18) completes the proof of 1°.

When x_0 is the right endpoint of I, the argument is similar.

Note, that having already proved (10) for the endpoints of I, in the same way as in the previous part of the proof of 2° we can also prove (12) for the endpoints. The proof is completed. □

Remark 2. Lemma 3 remains true if we replace (11) by

$$\varphi_k \leqslant \varphi_{k+1}, \quad k \in \mathbb{N}.$$

3. Main results

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval. A local operator K maps $CM_+(I)$ into itself if, and only if, there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ continuous in both variables and nondecreasing with respect to each variable such that, for all $\varphi \in CM_+(I)$,*

$$K(\varphi)(x) = h\bigl(x, \varphi(x)\bigr), \quad x \in I. \tag{19}$$

Proof. Assume first that a local operator K maps $CM_+(I)$ into itself. We begin with the construction of h. For an arbitrary $y_0 \in \mathbb{R}$ let us define a function $P_{y_0} : I \to \mathbb{R}$ by

$$P_{y_0}(x) := y_0, \quad x \in I. \tag{20}$$

Of course P_{y_0}, as a constant function, belongs to $CM_+(I)$. To construct the function $h : I \times \mathbb{R} \to \mathbb{R}$, fix arbitrarily $x_0 \in I$, $y_0 \in \mathbb{R}$ and put

$$h(x_0, y_0) := K(P_{y_0})(x_0). \tag{21}$$

Since for all functions φ,

$$\varphi(x_0) = P_{\varphi(x_0)}(x_0), \tag{22}$$

by (10) and (21), we have

$$K(\varphi)(x_0) = K(P_{\varphi(x_0)})(x_0) = h\bigl(x_0, \varphi(x_0)\bigr). \tag{23}$$

To prove the uniqueness of h assume that $g : I \times \mathbb{R} \to \mathbb{R}$ is such that

$$K(\varphi)(x) = g\bigl(x, \varphi(x)\bigr)$$

for all $\varphi \in CM_+(I)$ and $x \in I$. To show that $g = h$ let us fix arbitrarily $x \in I$, $y \in \mathbb{R}$ and take $\varphi \in CM_+(I)$ being a constant function y.

Now, by (23), we have

$$g(x, y) = g\bigl(x, \varphi(x)\bigr) = K(\varphi)(x) = h\bigl(x, \varphi(x)\bigr) = h(x, y),$$

which proves the uniqueness of h and (19).

To prove that h is nondecreasing with respect to the first variable, fix $y_0 \in \mathbb{R}$, take arbitrary $x_1, x_2 \in I$, $x_1 < x_2$, and the function $P_{y_0} : I \to \mathbb{R}$ defined by (20). According to (21) we have

$$h(x_1, y_0) - h(x_2, y_0) = K(P_{y_0})(x_1) - K(P_{y_0})(x_2)$$

and, by the monotonicity of $K(P_{y_0})$, we immediately get

$$h(x_1, y_0) - h(x_2, y_0) \leqslant 0.$$

To show that h is nondecreasing with respect to the second variable, fix $x_0 \in I$, take $y_1, y_2 \in \mathbb{R}$, $y_1 < y_2$ and define $P_{y_1} : I \to \mathbb{R}$ and $P_{y_2} : I \to \mathbb{R}$ by

$$P_{y_1}(x) := y_1, \qquad P_{y_2}(x) := y_2, \quad x \in I.$$

Hence and from (21) we get

$$h(x_0, y_1) - h(x_0, y_2) = K(P_{y_1})(x_0) - K(P_{y_2})(x_0)$$

and, consequently, by Lemma 1,

$$K(P_{y_1})(x_0) - K(P_{y_2})(x_0) \leqslant 0,$$

which completes the proof that h is nondecreasing with respect to both variables on $I \times \mathbb{R}$.

The proof of the continuity of h is divided into the following five steps:

1° for every $(x_0, y_0) \in I \times \mathbb{R}, x_0 \neq \sup I$ and for every two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ convergent to x_0 and y_0, respectively, such that

$$x_n \in I, \qquad x_{n+1} < x_n, \qquad x_n > x_0, \quad n \in \mathbb{N}, \tag{24}$$

$$y_n \in \mathbb{R}, \qquad y_{n+1} \leqslant y_n, \qquad y_n \geqslant y_0, \quad n \in \mathbb{N}, \tag{25}$$

we have

$$\lim_{n\to\infty} h(x_n, y_n) = h(x_0, y_0); \tag{26}$$

2° for every $(x_0, y_0) \in I \times \mathbb{R}$, $x_0 \neq \inf I$ and for every increasing sequence $(x_n)_{n\in\mathbb{N}}$ and nondecreasing sequence $(y_n)_{n\in\mathbb{N}}$ convergent to x_0 and y_0, respectively, (26) is fulfilled;

(in particular, 1° and 2° show that h is continuous with respect to the first variable);

3° h is continuous with respect to the second variable;

4° for every $(x_0, y_0) \in I \times \mathbb{R}$ and for every two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ convergent to x_0 and y_0, respectively, such that

$$x_n \in I, \qquad x_n < x_{n+1}, \qquad x_n < x_0, \quad n \in \mathbb{N}, \tag{27}$$

$$y_n \in \mathbb{R}, \qquad y_{n+1} < y_n, \qquad y_0 < y_n, \quad n \in \mathbb{N}, \tag{28}$$

the equality (26) is fulfilled;

5° for every $(x_0, y_0) \in I \times \mathbb{R}$, $x_0 \neq \sup I$ and for every $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ convergent to x_0 and y_0, respectively, such that

$$x_n \in I, \qquad x_{n+1} < x_n, \qquad x_n > x_0, \quad n \in \mathbb{N}, \tag{29}$$

$$y_n \in \mathbb{R}, \qquad y_n < y_{n+1}, \qquad y_n < y_0, \quad n \in \mathbb{N}, \tag{30}$$

the equality (26) is fulfilled.

To prove 1° fix arbitrarily $x_0 \in I$, $x_0 \neq \sup I$, $y_0 \in \mathbb{R}$ and take two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ which satisfy (24) and (25), convergent to x_0 and y_0, respectively. By Lemma 2 there exists a function $\gamma \in CM_+(I)$ such that $\gamma(x_n) = y_n$ for all $n \in \mathbb{N}_0$. Thus, by (23), we have

$$h(x_n, y_n) - h(x_0, y_0) = h\big(x_n, \gamma(x_n)\big) - h\big(x_0, \gamma(x_0)\big) = K(\gamma)(x_n) - K(\gamma)(x_0)$$

and, applying the continuity of $K(\gamma)$, we get (26).

By Remark 2, step 2° may be proved in a similar way as 1°.

To prove 3° fix arbitrarily $x_0 \in I$, $y_0 \in \mathbb{R}$. First we show that

$$\lim_{y\to y_0^+} h(x_0, y) = h(x_0, y_0). \tag{31}$$

To this end take an arbitrary real sequence $(y_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} y_n = y_0, \qquad y_n > y_0, \quad n \in \mathbb{N},$$

and choose a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} y_{n_k} = y_0, \qquad y_{n_k} > y_0, \qquad y_{n_{k+1}} \leqslant y_{n_k}, \quad k \in \mathbb{N}. \tag{32}$$

Define the constant functions $P_{y_{n_k}} : I \to \mathbb{R}$ by the formula

$$P_{y_{nk}}(x) = y_{n_k}, \quad x \in I.$$

Since $(P_{y_{n_k}})_{k\in\mathbb{N}}$ converges uniformly to P_{y_0}, by (21), (32) and Lemma 3, we obtain

$$\lim_{k\to\infty} h(x_0, y_{n_k}) = \lim_{k\to\infty} K(P_{y_{n_k}})(x_0) = K(P_{y_0})(x_0) = h(x_0, y_0),$$

which proves (31).

Similarly, by Remark 2, we can show

$$\lim_{y\to y_0^-} h(x_0, y) = h(x_0, y_0),$$

which completes the proof of 3°.

To prove 4° suppose, contrary to our claim, that there exists a real $\varepsilon > 0$, $(x_0, y_0) \in I \times \mathbb{R}$ and two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ convergent to x_0, y_0, respectively, which satisfy (27) and (28) such that $|h(x_n, y_n) - h(x_0, y_0)| \geqslant \varepsilon$.

Suppose first that

$$h(x_n, y_n) - h(x_0, y_0) \geqslant \varepsilon. \tag{33}$$

By 3° there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $n \in \mathbb{N}$,

$$h(x_0, y_n) - h(x_0, y_0) < \varepsilon.$$

Hence and by (33) we get

$$h(x_n, y_n) \geqslant h(x_0, y_0) + \varepsilon > h(x_0, y_n), \quad n \geqslant n_0,\ n \in \mathbb{N},$$

which contradicts the fact that h is nondecreasing with respect to the first variable.

Suppose now that

$$h(x_n, y_n) - h(x_0, y_0) \leqslant -\varepsilon, \quad n \geqslant n_0,\ n \in \mathbb{N}. \tag{34}$$

Taking into account the continuity of h with respect to the first variable, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $n \in \mathbb{N}$,

$$h(x_0, y_0) - h(x_n, y_0) < \varepsilon.$$

Hence and by (34) we get

$$h(x_n, y_n) \leqslant h(x_0, y_0) - \varepsilon < h(x_n, y_0), \quad n \geqslant n_0,\ n \in \mathbb{N},$$

which contradicts the fact that h is nondecreasing with respect to the second variable.

By a similar reasoning we can show 5°.

To prove the converse implication suppose that there exists a continuous and nondecreasing function $h : I \times \mathbb{R} \to \mathbb{R}$ such that (19) is fulfilled and take an arbitrary $\varphi \in CM_+(I)$. We shall show that $K(\varphi) \in CM_+(I)$.

To this end fix arbitrarily $x_1, x_2 \in I$, $x_1 < x_2$. By (19) and the monotonicity of φ we have

$$K(\varphi)(x_1) = h\big(x_1, \varphi(x_1)\big) \leqslant h\big(x_1, \varphi(x_2)\big) \leqslant h\big(x_2, \varphi(x_2)\big) = K(\varphi)(x_2),$$

which shows that $K(\varphi) \in M_+(I)$.

To show the continuity of $K(\varphi)$, fix $(x_0, y_0) \in I \times \mathbb{R}$ and assume that $x_n \in I$, $n \in \mathbb{N}$, is a sequence converging to x_0. Thus, by the continuity of φ and h,

$$\lim_{n\to\infty} K(\varphi)(x_n) = \lim_{n\to\infty} h\big(x_n, \varphi(x_n)\big) = h\big(x_0, \varphi(x_0)\big) = K(\varphi)(x_0),$$

which proves that $K(\varphi) \in C^0(I)$.

As, obviously, K is locally defined, the proof is completed. □

Definition. Let $X \subset \mathbb{R}$ and a function $h : X \times \mathbb{R} \to \mathbb{R}$ be fixed. An operator $K : \mathbb{R}^X \to \mathbb{R}^X$ given by

$$K(\varphi)(x) = h\big(x, \varphi(x)\big), \quad \varphi \in \mathbb{R}^X\ (x \in X),$$

is said to be the Nemytskii (or superposition) operator. The function h is referred to as the generator of the operator K.

Corollary 1. *A local operator K maps $CM_+(I)$ into itself if, and only if, it is a Nemytskii (superposition) operator of the continuous in both variables and nondecreasing with respect to each variable generator.*

Remark 3. The assumption that the domain and the range of the considered operator is contained in $C^0(I)$ is essential. To show it consider the following

Example. Let $K : M_+(I) \to M_+(I)$ is defined by the formula

$$K(\varphi)(x) = \begin{cases} \varphi(x-) & \text{for } x \neq \inf I,\ x \in I, \\ \varphi(x) & \text{for } x = \inf I. \end{cases}$$

It is clear that K is locally defined, but it is not of the form $K(\varphi)(x) = h(x, \varphi(x))$ for any $h : I \times \mathbb{R} \to \mathbb{R}$.

4. The results for three remaining cases

A simple application of Theorem 1 allows to characterize the local operators in the three remaining cases. We begin with the case when $K : CM_-(I) \to CM_-(I)$.

Theorem 2. *Let $I \in \mathbb{R}$ be an interval. A local operator K maps $CM_-(I)$ into itself if, and only if, there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ continuous in both variables, nonincreasing with respect to the first variable and nondecreasing with respect to the second variable such that, for all $\varphi \in CM_-(I)$,*

$$K(\varphi)(x) = h\bigl(x, \varphi(x)\bigr), \quad x \in I.$$

Proof. Let $K : CM_-(I) \to CM_-(I)$ be a local operator. Define $K_1 : CM_+(I) \to CM_+(I)$ by

$$K_1(\varphi) := -K(-\varphi), \quad \varphi \in CM_+(I)\ (x \in I). \tag{35}$$

As K_1 is local, by Theorem 1, there exists a unique function $h_1 : I \times \mathbb{R} \to \mathbb{R}$ which is continuous in both variables, nondecreasing with respect to each variable and, for all $\varphi \in CM_+(I)$,

$$K_1(\varphi)(x) = h_1\bigl(x, \varphi(x)\bigr), \quad x \in I.$$

Put

$$h(x, y) := -h_1(x, -y).$$

Hence, taking into account (35) and replacing φ by $-\varphi$, we obtain

$$K(\varphi)(x) = -K_1(-\varphi)(x) = -h_1\bigl(x, -\varphi(x)\bigr) = h\bigl(x, \varphi(x)\bigr) \quad (x \in I),$$

for all $\varphi \in CM_-(I)$. As the required properties of the function h are evident, this completes the proof. $\square$

Corollary 2. *A local operator K maps $CM_-(I)$ into itself if, and only if, it is a Nemytskii operator of the continuous in both variables, nonincreasing with respect to the first variable and nondecreasing with respect to the second variable generator.*

In a similar way we obtain the following

Theorem 3. *An operator K is (CM_+, CM_-)-local if, and only if, there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ continuous in both variables and nonincreasing with respect to each variable such that, for all $\varphi \in CM_+(I)$,*

$$K(\varphi)(x) = h\bigl(x, \varphi(x)\bigr), \quad x \in I.$$

Theorem 4. *An operator K is (CM_-, CM_+)-local if, and only if, there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ continuous in both variables, nondecreasing with respect to the first variable and nonincreasing with respect to the second variable such that, for all $\varphi \in CM_-(I)$,*

$$K(\varphi)(x) = h\bigl(x, \varphi(x)\bigr), \quad x \in I.$$

Corollary 3. *An operator K is (CM_+, CM_-)-local or (CM_-, CM_+)-local if, and only if, it is a Nemytskii operator of the continuous and suitably monotone in both variables generator.*

We end with the following

Remark 4. Let $I \subset \mathbb{R}$ be an interval.

1° If an operator K is (CM_+, CM_-)-local and (CM_-, CM_-)-local then, obviously, K is constant, that is there is a function $b \in CM_-(I)$ such that

$$K(\varphi) = b, \quad \varphi \in CM_+(I) \cup CM_-(I).$$

2° If an operator K is (CM_+, CM_+)-local and (CM_-, CM_-)-local then

$$K(\varphi) = h \circ \varphi, \quad \varphi \in CM_+(I) \cup CM_-(I),$$

for some nondecreasing continuous function $h : \mathbb{R} \to \mathbb{R}$.

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